CHAPTER 3

What Are Generalizations in Mathematics?

What is π? Often students cannot explain what π is, stating it is 3.14 or 22/7. In fact, π is the ratio of the circumference to the diameter of *every and any* circle. What are the trigonometric ratios? A common explanation is that trigonometric ratios are buttons on a calculator, or they are sine, cosine, or tangent of an angle as formulae to be memorized. In fact, these three trigonometric ratios connect ratios of sides of similar right-angled triangles, which help to solve real-life problems in surveying, architecture, and astronomy.

Generalizations give us explanations of why and what we want our students to comprehend in terms of the relationship between two or more concepts. Also known in education circles as *enduring understandings, essential understandings,* or *big ideas* of the unit, they summarize what we would like our students to take away after their unit of study. Liz Bills and colleagues (2007) explain, "Provoking generalisation is more about releasing learners' natural powers than it is about trying to force feed. Because promoting mathematical generalisation lies at the core of all mathematics teaching, at all ages, and because it concerns the development of higher psychological processes that are most likely to be accessible to learners if they are in the presence of someone more expert displaying disposition to and techniques for generalising, it is important for teachers to be seen to generalise, to value learner's attempts at generalisation, and to get out of the learner's way so that they can generalise for themselves" (p. 56).

Generalizations provide clear goals and align with the National Council of Teachers of Mathematics (NCTM, 2014) Mathematics Teaching Practices:

> Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are

learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

What Is the Difference Between a Generalization and a Principle in Mathematics?

Lynn Erickson (2014) distinguishes between generalizations and principles in mathematics by explaining that "Generalizations are defined as two or more concepts stated as a sentence of relationship. They are understandings that transfer through time, across cultures, and across situations. Generalizations contain no proper nouns, past tense verbs, or pronouns that would associate the idea with a particular person or group . . . . Generalizations are truths supported by factual examples, but they may include a qualifier (*often, can, may*) when the idea is important but does not hold in *all* instances" (p. 34).

Generalizations are the overriding ideas we would like our students to understand and transfer that may contain a qualifier such as *may* or *can*. Because mathematics is a conceptual language, generalizations are statements of two or more concepts that transfer through time, cultures, and across situations in the context of mathematics. For example, the angle sum in triangles *may* add to 180 degrees. In Euclidean geometry (the study of "flat space"), triangles always add up to 180 degrees. However, if we look at non-Euclidean geometry, this generalization does not hold true, so the qualifier *may* is used. Mathematical generalizations can also be expressed with an appropriate domain, so a qualifier is not needed.

Generalizations provide goals when unit planning and help learners to construct and make connections with what they are learning. Through inductive, inquiry-led teaching, learners are guided to discover generalizations and answer the question, "Why am I learning this and what is the point beyond this example?"

Wiggins and McTighe (2006a) talk similarly about statements of understanding, which they refer to as "enduring understandings": "In UbD (Understanding By Design), designers are encouraged to write [enduring understandings] as full sentence statements, describing what, specifically, students should understand about the topic. The stem "Student will understand that . . ." provides a practical tool for identifying these understandings" (p. 342).

The term *enduring understandings* is the same as Generalizations and Principles in the Structures of Knowledge and Process. There are many terms that different schools use to identify these statements of conceptual relationship: generalizations, essential understandings, enduring understandings, or statements of inquiry. Wiggins and McTighe (2006b) define two types of enduring understandings: *overarching* and *topical*.

Overarching generalizations are also known as Statements of Inquiry in the IB Middle Years math program. They give us the breadth of knowledge in a unit of work while topical generalizations give us the depth of knowledge in a unit of work.

Overarching enduring understandings (**overarching generalizations**) are understandings beyond the specifics of a unit. They are the larger, transferable insights we want students to acquire. They often reflect year-long program understandings.

Topical understandings (**topical generalizations**) are subject or topic specific. They focus on a particular insight we want students to acquire within a unit of study. Figure 3.1 is an example of overarching and topical enduring understandings, adapted from Wiggins and McTighe (2006b, p. 114).

This chapter will discuss topical enduring understandings, which will be referred to as *topical generalizations,* and how to craft topical generalizations for mathematics.

Principles in Structures of Knowledge and Process are laws or foundational truths that hold all the attributes of generalizations and commonly describe real-life situations. Theorems are the "principles" in mathematics. Principles never contain qualifiers such as *may*, *can,* or *often*. Mathematical theorems, such as the Pythagorean theorem, Fermat's theorem, and Pick's theorem, are all principles when written out as statements of relationship in either the Structure of Knowledge or the Structure of Process. Some theorems are derived from observations around the world and are considered the laws of mathematics.

FIGURE 3.1: THE TWO TYPES OF ENDURING UNDERSTANDINGS: OVERARCHING AND TOPICAL

Overarching Enduring Understanding	Topical Enduring Understanding
Mathematics reveals patterns that might have remained unseen.	Statistical analysis and graphic displays often reveal patterns in seemingly random data or populations, enabling predictions.

Examples of Mathematics Generalizations

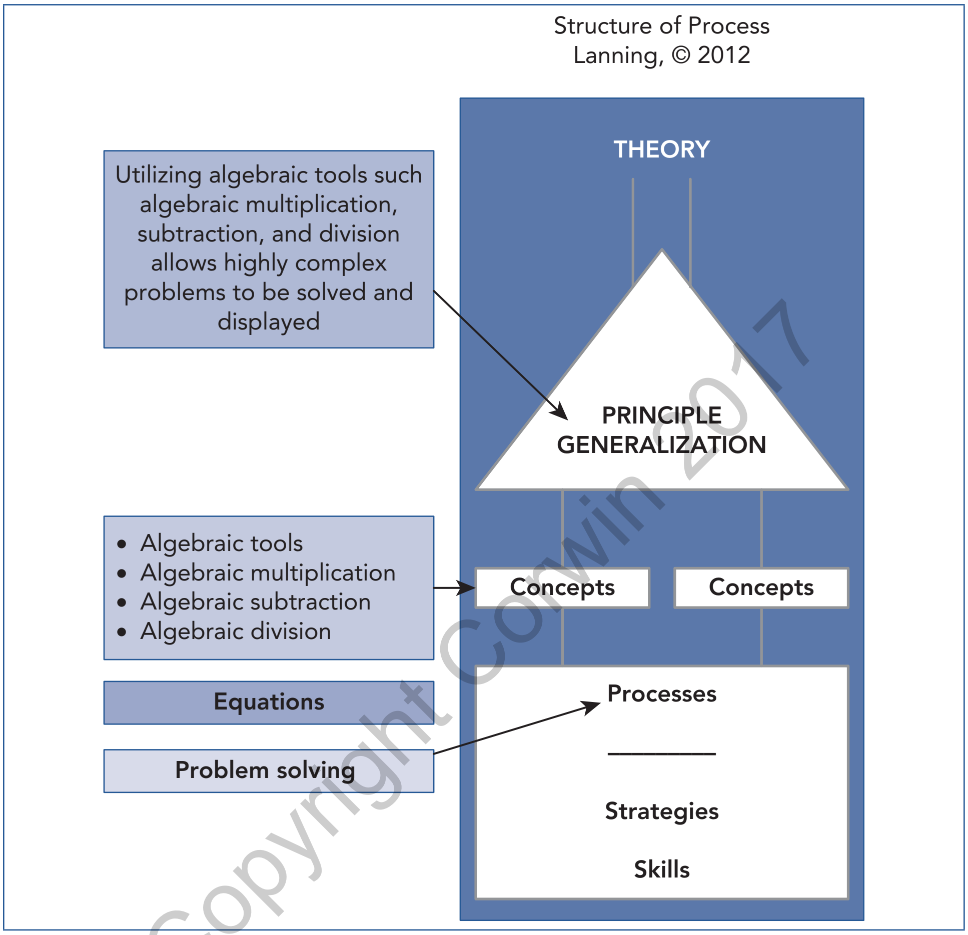
Let's look at some specific examples of mathematics generalizations and principles and how they are supported by the concepts of the unit, processes, and facts.

Generalization Example 1

Utilizing algebraic tools such algebraic multiplication, subtraction, and division allow highly complex problems to be solved and displayed.

The concepts of *algebraic tools, multiplication, subtraction,* and *division* support the preceding generalization. This is a process generalization that supports the process of problem solving and can be represented in the Structure of Process as shown in Figure 3.2.

FIGURE 3.2: THE STRUCTURE OF PROCESS FOR EQUATIONS



Adapted from original Structure of Process figure from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.

Generalization Example 2

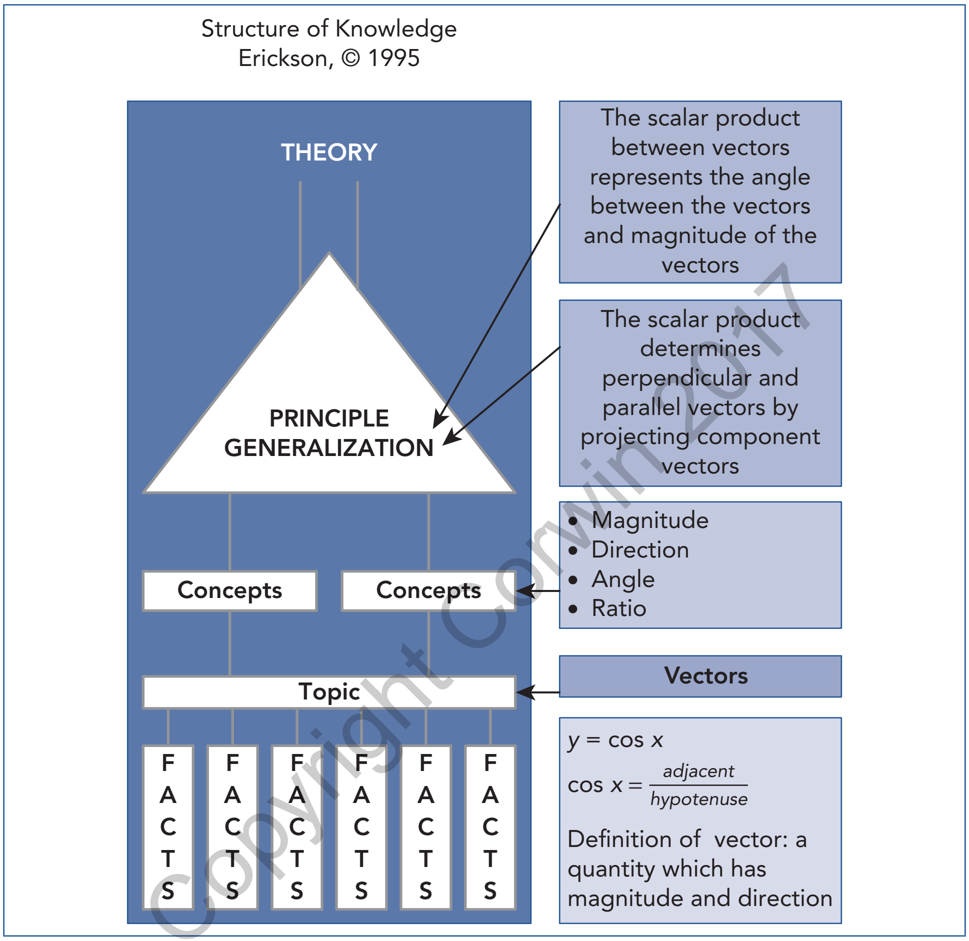
The following generalizations refer to the meso concept *vectors.* The scalar product is used to work out magnitudes and angles between two vectors.

The scalar product between vectors represents the angle between the vectors and magnitude of the vectors.

Scalar product determines perpendicular and parallel vectors by projecting component vectors.

These generalizations are supported by the fact $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$ and are supported by the concepts of *magnitude, direction, angle, perpendicular, parallel,* and *component vector,* which can be represented diagrammatically in the Structure of Knowledge, as shown in Figure 3.3.

FIGURE 3.3: THE STRUCTURE OF KNOWLEDGE FOR VECTORS



Adapted from original Structure of Knowledge figure from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.

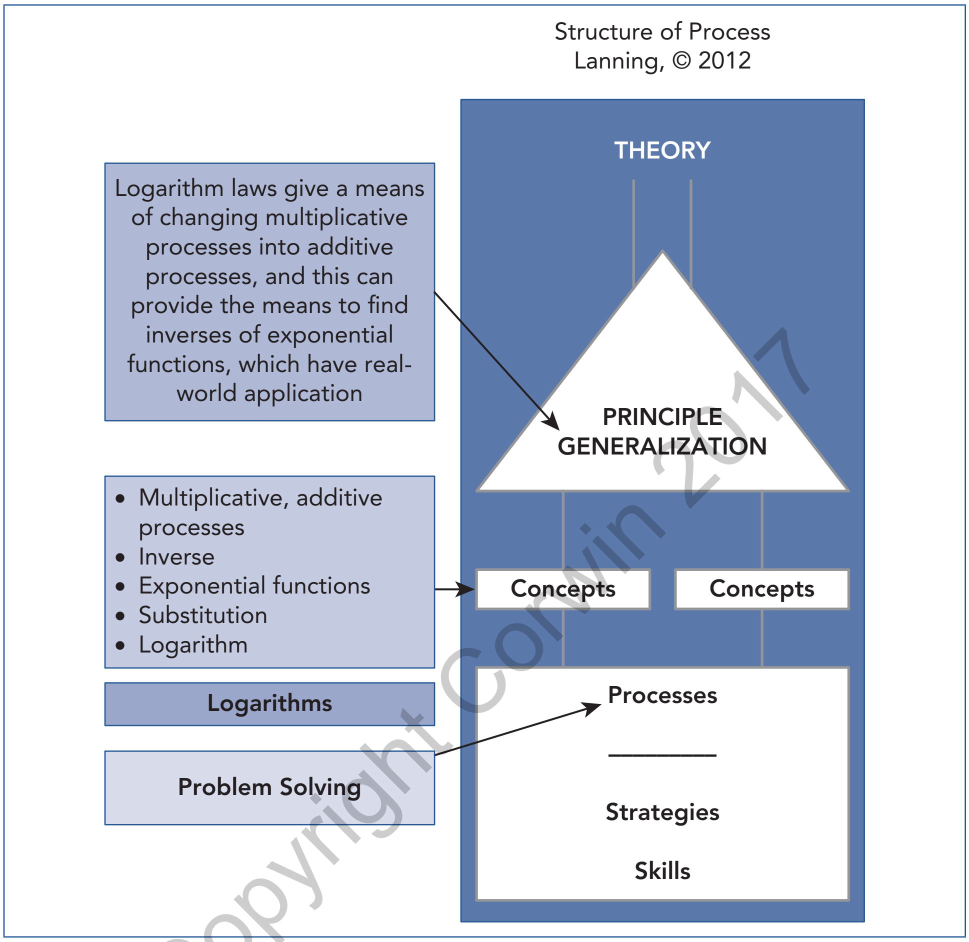
Generalization Example 3

Many students find the concept of logarithms challenging. Students often memorize logarithmic laws with no understanding of their purpose. Understanding that logarithms are inverses of exponential functions and that exponential functions represent continuous compounded growth shows deep intellectual depth.

> Logarithm laws give a means of changing multiplicative processes into additive processes, and this can provide the means to find inverses of exponential functions, which represent continuous compounded growth.

The two mathematical processes, *problem solving* and *making connections,* support this process generalization, as shown in Figure 3.4.

FIGURE 3.4: THE STRUCTURE OF PROCESS FOR LOGARITHMS



Adapted from original Structure of Process figure from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.

Generalization Example 4

The following generalizations summarize the concepts and significance of *finding the roots* (using the quadratic formula) and *discriminant* in quadratic equations.

The expression underneath the square root in the quadratic formula, the discriminant, determines the nature of the roots, which highlight geometrical features.

The quadratic formula determines the zeros of functions, the roots of equations, and the *x*-intercepts graphically.

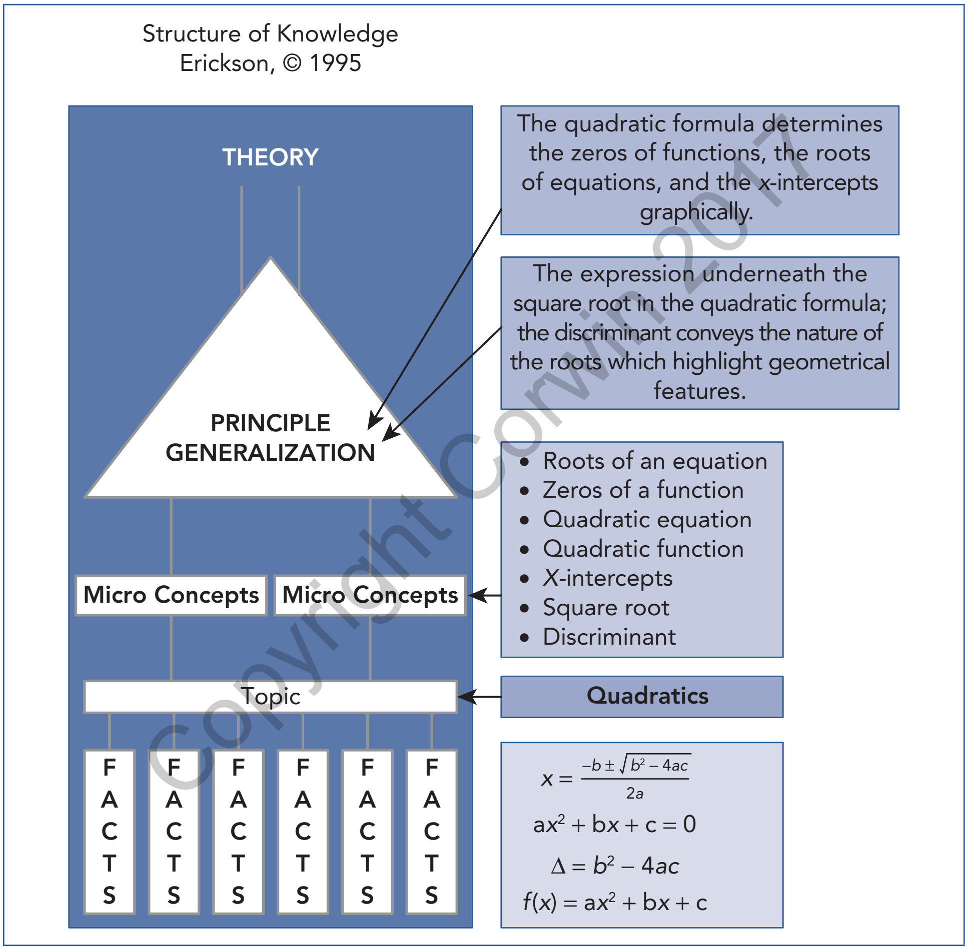
These generalizations are supported by the following fact (mathematical formula):

For $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

These generalizations support the concepts of *equation, zeros, roots,* and *quadratic function* and can be represented in the Structure of Knowledge as shown in Figure 3.5.

FIGURE 3.5: THE STRUCTURE OF KNOWLEDGE FOR QUADRATICS



Adapted from original Structure of Knowledge figure from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.

Generalization Example 5

Figure 3.6 depicts the Structure of Knowledge and Structure of Process side by side for the meso concept *quadratics*. The micro concepts on the Structure of Knowledge are *equation, zeros, roots,* and *quadratic function*. These micro concepts support these two generalizations:

The quadratic formula determines the zeros of equations, roots of functions, and the *x*-intercepts graphically.

The expression underneath the square root in the quadratic formula, the discriminant, conveys the nature of the roots, which highlight geometrical features.

For the Structure of Process, the micro concepts are drawn from the process of creating representations in addition to following strategies and skills, such as using an *xy*-plane, using a table or graphs, and utilizing substitution. These micro concepts support the following generalization:

Creating visual depictions of a problem using different modes of representation (graphs, tables, etc.) helps explain the problem and reveal a solution.

The interplay between both these structures support learning and conceptual understanding in this unit. In other words, these structures support the understanding of the generalizations of a unit.

How Do We Craft Quality Mathematics Generalizations?

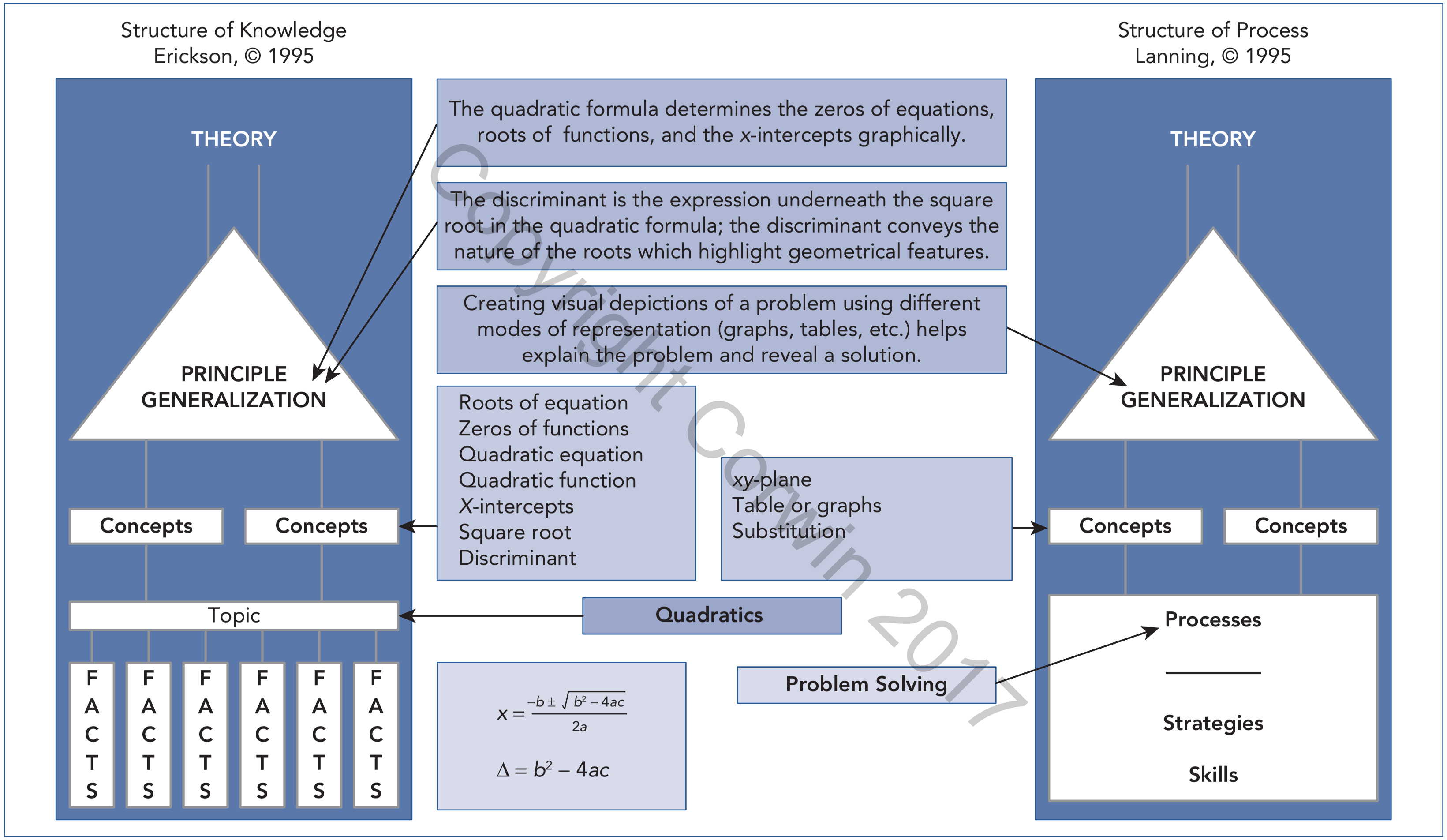
Utilizing quality generalizations when planning a unit of work guides the work of a concept-based teacher. Generalizations and principles in unit plans provide a clear focus for curriculum and instruction. Giving plenty of opportunities for students to generalize is the essence to learning mathematics. All concepts in mathematics can be expressed in a sentence of relationship in a variety of ways, according to factors such as student level and the teacher's aim. Because of this, the writing of generalizations should be a collaborative process within a faculty and part of the planning for a concept-based unit. A unit of work may typically have six to eight generalizations, with five to six instructional units over a year.

When starting to write generalizations, it is common to write statements that are too general. The verbs *is, are, have, impacts, affects* and *influences* are overused and create weak statements.

Avoiding certain "no no" verbs will create stronger statements of conceptual understanding.

Lynn Erickson (2007) provides a three-step guide to help scaffold the process of writing quality generalizations and advises avoiding the use of the verbs *is, are, have, impacts, affects,* and *influences,* referring to these as "no no verbs."

FIGURE 3.6: SIDE BY SIDE: THE STRUCTURE OF KNOWLEDGE AND THE STRUCTURE OF PROCESS FOR QUADRATICS



Adapted from original Structure of Knowledge and Structure of Process figures from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.

On the companion website you will find a table of "no no" verbs (Figure M2.1), two examples of how to scaffold generalizations (Figures M2.2 and M2.3), a template of the scaffolding process when writing generalizations (Figure M2.4), a comprehensive list of sample verbs to use when crafting generalizations (Figure M2.5), and a checklist for crafting generalizations (Figure M2.6).

When writing your own generalizations, try using the stem "Students will understand that . . ." to help focus on what students should understand as opposed to what students will know and do in terms of skills. Here is an example of the three-step guide of generalizations for the topic of quadratics:

Level 1 Students will understand that . . .

The quadratic formula is one method to solve a quadratic equation.

Level 2 How or why?

The quadratic formula describes the roots or zeros of the function and helps solve a quadratic equation.

Level 3 So what is the significance or effect?

The expression underneath the square root in the quadratic formula, the discriminant, conveys the nature of the roots, which highlight geometrical features.

The Level 2 generalization is the teaching target and where we focus instruction.

Figure 3.7 summarizes the scaffolding process when writing quality generalizations.

FIGURE 3.7: SCAFFOLDING TEMPLATE

Level 1	Students will understand that . . .
Level 2	How or why? (Choose which question is most appropriate)
Level 3	So what? What is the significance or effect?

Writing quality generalizations takes practice and is a skill that can be mastered. The teaching target is the Level 2 generalization; use Level 3 for extension or clarification.

We use Level 1 generalizations to show how to avoid "no no" verbs. Our goal is to write generalizations at Level 2 and skip Level 1.

Consider the following generalization:

Level 1: The coefficients of a quadratic function in a connected equation affect the quadratic formula.

Level 1 generalizations are weak and do not help learners to fully understand what and why they are learning. They often include "no no" verbs such as *is*, *are, have, impact, affect,* and *influence*. To help learners understand what they are learning, ask the questions "How?" or "Why?" to help scaffold a generalization to the next level. Scaffolding gives learners greater clarity and conceptual specificity, which results in depth of understanding.

The above Level 1 generalization becomes a Level 2 generalization:

Level 2: The quadratic formula utilizes the coefficients of a quadratic function in a connected equation and describes the roots of a quadratic function.

Level 1 generalizations should be deleted once the scaffolding process offers a stronger Level 2 generalization.

Level 3 generalizations address the question, "So what is the significance or effect?" To scaffold further to extend our students' learning, we ask the question, "So what?" which explains the purpose of the generalization. It is important to note, however, that in mathematics the critical understanding is usually expressed in Level 2. Level 3 can be used to express the real-world value of mathematics or to mathematically extend the Level 2 understanding with greater conceptual specificity, if appropriate.

Level 3: Different methods of solving quadratic equations distinguish the roots of the quadratic equation, which contribute to providing a graphical representation of the function, which, in turn, describes real-life problems.

or

Level 3: The expression underneath the square root in the quadratic formula, the discriminant, conveys the nature of the roots, which highlight geometrical features.

Figures 3.8 and 3.9 show some more examples of generalizations for the topic of sequences and series and logarithms.

FIGURE 3.8: SCAFFOLDING GENERALIZATION FOR SEQUENCES AND SERIES

Level 1	Students will understand that . . . *Arithmetic and geometric sequences and series are structured by a set of formulae and definitions.* This generalization contains a "passive voice" verb, which weakens the sentence. Flipping the sentence emphasizes the strong verb for a Level 2 generalization.
Level 2	How or why? *A set of formulae and definitions structure arithmetic and geometric sequences.* *Whether arithmetic and geometric sequences and series share a common difference or a common ratio distinguish one from another.*
Level 3	So what? What is the significance or effect? *Arithmetic and geometric sequences and series describe patterns in numbers and supply algebraic tools that help to solve real-life situations.*

Teaching target is Level 2 generalizations

FIGURE 3.9: SCAFFOLDING A PROCESS GENERALIZATION

Level 1	Students will understand that . . . *Logarithm laws affect logarithmic expressions.*
Level 2	How or why? *Logarithm laws reduce logarithmic expressions by utilizing the inverse process of exponential functions, which represent continuous compounded growth.*
Level 3	So what? What is the significance or effect? *Logarithm laws provide a means of changing multiplicative processes into additive processes, and this can provide the means to find inverses of exponential functions (continuous compounded growth), which have real-world applications.*

In a particular unit of work, not all generalizations will move toward Level 3, as Level 2 generalizations provide strong, clear statements of conceptual understandings. Ensure that your generalizations include the key concepts of the unit. Remember the focus of your teaching is on Level 2 generalizations.

I used to think that generalizations were statements that were not necessarily meaningful for students as they contained the big ideas with little significance. I now know that generalizations can represent significant, enduring understandings that I want my students to be able to understand and communicate. Avoiding the use of weaker "no no" verbs allows us to write stronger generalizations, which then gave me my moment of epiphany about the importance of principles and generalizations!

John Edwards, Head of History
King George V School, Hong Kong

The companion website provides a comprehensive list of sample verbs to use when crafting generalizations (Figure M2.5).

How Do We Draw Out Conceptual Understandings From Our Students?

Effective mathematics learning arises out of guiding students to particular principles or generalizations through inductive inquiry.

Key to inductive teaching is providing students with specific examples from which to draw generalizations. Traditional curriculum and instruction have focused on deductive approaches: telling students the generalization and then asking students to study specific examples. Concept-based curricula emphasize inductive approaches guiding students, through inquiry, to discover generalizations and principles for themselves.

Figure 3.10 shows an inductive, inquiry-led activity guiding students to understand the following generalization:

The expression underneath the square root in the quadratic formula, the discriminant, conveys the nature of the roots, which highlight geometrical features of the quadratic function.

When using inductive approaches, students need to be given opportunities to form conceptual understandings from their learning experiences by writing in mathematics. The learning experiences should involve students looking at specific examples and seeking patterns in order to devise and write generalizations.

In Figure 3.10, the prompt "Explain in your own words the significance of the expression underneath the square root sign in the quadratic formula and how this

Sample Student Learning Experience

FIGURE 3.10: AN EXAMPLE OF INDUCTIVE INQUIRY TO DRAW A GENERALIZATION

Roots of Quadratic Equations

Look at the four examples of quadratic equations and solve them using the quadratic formula.

1. $3x^2 - x - 1 = 0$
2. $2x^2 - 3x - 5 = 0$
3. $4x^2 - 12x + 9 = 0$
4. $x^2 - x + 1 = 0$

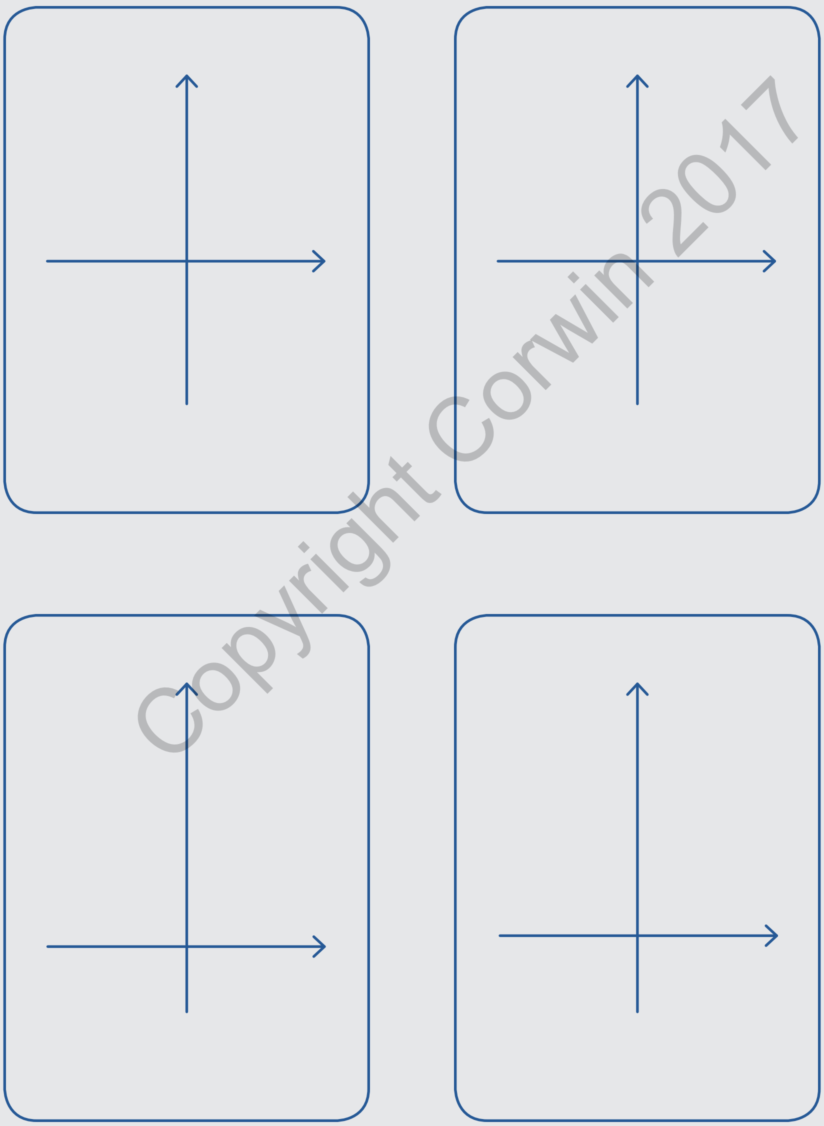
What does solving the above quadratic equations tell you about the associated quadratic functions and their graphs? What is the special name for these values?

What do you notice about what is underneath the square root sign in the four examples above?

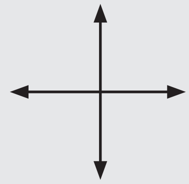
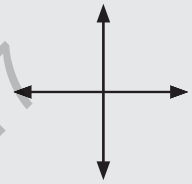
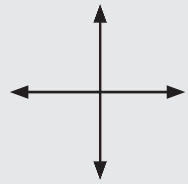
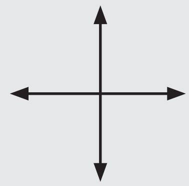
Find out what the "*bit*" underneath the square root is called and include an explanation.

Draw a sketch of these four parabolas.

1. $y = 3x^2 - x - 1$
2. $y = 2x^2 - 3x - 5$
3. $y = 4x^2 - 12x + 9$
4. $y = x^2 - x + 1$



Complete this table:

Value of the ________	Nature of the Roots	Graphical Sketch
$b^2 - 4ac > 0$ and a perfect square		
$b^2 - 4ac > 0$ and not a perfect square		
$b^2 - 4ac = 0$		
$b^2 - 4ac < 0$		

Do you understand the concept?

1. For each part, find the value of the ________ and state whether the equation will have 2 roots, 1 repeated root, or no roots, and include a sketch.

(a) $x^2 + 11x - 2 = 0$

(b) $x^2 - 3x + 3 = 0$

(c) $9x^2 + 6x + 1 = 0$

(d) $x^2 + 13x + 36 = 0$

2. If $2x^2 + bx + 50 = 0$ has a repeated root, find the value of b. Include a sketch.

3. Find the range of values k can take for $9x^2 + kx + 4 = 0$ to have 2 real distinct roots.

Explain in your own words the significance of the expression underneath the square root sign in the quadratic formula and how this helps us with the geometrical significance of a parabola.

For a completed version of Figure 3.10, please visit the companion website.

helps us with the geometrical significance of a parabola" is used to draw the generalization from students. Research shows that encouraging students to write in the math classroom develops the ability to organize, understand, analyze, and reflect on the new learning (Pugalee, 2005; Stonewater, 2002). Writing allows students to communicate how they have processed information and skills and whether they possess the conceptual understandings of a unit. David Sousa (2015) affirms, "Adding this [writing in math] kinesthetic activity engages more brain areas and helps students to organize their thoughts about the concept" (p. 201).

Graphic organizers are another useful tool for helping draw out generalizations. A graphic organizer can give students a visual representation of how the concepts are related, and they will then be able to form generalizations.

Figure 3.11 is an example of a graphic organizer used to draw generalizations about trigonometry.

Students should be given the opportunity to explore or identify generalizations from their program of study. There are several strategies to help students with this process. Figure 3.12 lists ideas for drawing the generalizations from our students.

Learning experiences should be designed to draw out conceptual understandings, using correct mathematical terminology. If asked about the purpose of the unit, students need to be able to communicate the essence of the desired conceptual understanding verbally and nonverbally.

At the secondary school level, a year-long course may have five or six instructional units. Each unit may have six to eight generalizations to encourage guided inquiry, depending on the length of the unit.

It is important when identifying your conceptual understandings that thought goes into the planning of the entire unit and the concepts involved by utilizing a unit web, as suggested by Erickson and Lanning (2014). A unit webbing tool provides a strategy for including generalizations, which will be discussed in the next chapter. The next chapter will also discuss how to plan a unit of work, including guiding and essential questions to support inquiry leading to deep, conceptual understanding.

FIGURE 3.11: AN EXAMPLE OF A GRAPHIC ORGANIZER TO DRAW GENERALIZATIONS FROM STUDENTS FOR TRIGONOMETRY

What is the big idea?

Using the two strands—right-angled and non-right-angled triangles—write down micro concepts (ideas) in each strand. For each strand, write a generalization (statement) that connects the concepts and captures the most important idea.

What is the big idea?

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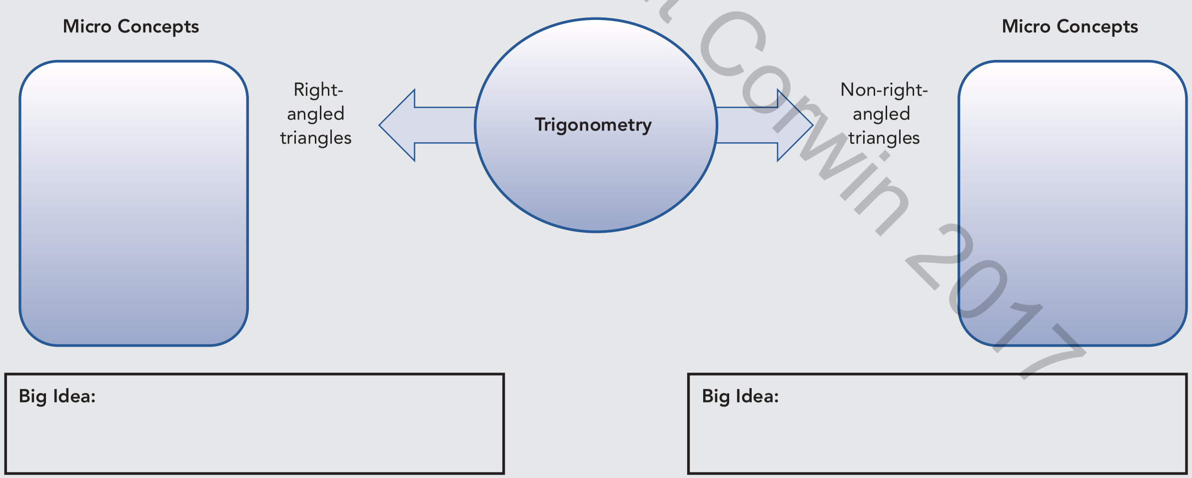

FIGURE 3.12: STRATEGIES TO DRAW GENERALIZATIONS FROM STUDENTS

Strategy	Example
Give the concepts and ask students to connect these concepts in a statement that reflects the purpose of their unit of study.	Write the following concepts in a related sentence. This statement should demonstrate your understanding of how these concepts are connected: zeros of a function, roots of an equation, *x*-intercepts, quadratic formula *or* parallel lines, gradient, transversal, angles
At the end of a lesson or unit of work ask students to complete the sentence, "I understand that . . ."	I understand that solving a quadratic equation by using the quadratic formula graphically displays the *x*-intercepts of a quadratic function.
Give students a choice of a few statements and ask students to choose one that explains the big idea of the lesson or unit and to justify why.	The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Solving a quadratic equation by using the quadratic formula graphically displays the *x*-intercepts of a quadratic function. The quadratic formula tells us what *x* is equal to and allows us to solve quadratic equations.
After a lesson or unit of study, ask students to create a concept map, which includes the concepts they have learned and how they are connected, with statements explaining them.	With the word *quadratics* at the center, create a concept map using the ideas you have learned and include statements to explain how they are connected.
Give students a cloze activity. Hint Jar: quadratic equation quadratic formula quadratic function *x*-intercepts	Solving a __________ ________ by using the _________ ________ graphically displays the __________ of a ______________.
Ask students to devise a headline that summarizes the key ideas (concepts) of a unit.	"SOHCAHTOA helps us to find angles and lengths of sides in right-angled triangles."

Chapter Summary

This chapter helps educators develop quality generalizations to foster students' conceptual understanding. Generalizations are statements of conceptual understanding and represent our goals for learning. They are also known as essential understandings, enduring understandings, or the big ideas in a unit of work.

There are two types of enduring understandings (generalizations): overarching and topical. This chapter discusses how to craft topical generalizations that are more specific to a unit of work and focus on particular insights we want students to acquire. In mathematics, generalizations describe the relationships between important concepts. Principles include mathematical theorems, which are cornerstone truths.

Lynn Erickson provides a three-step scaffolding process when writing generalizations and suggests avoiding certain overused verbs to create stronger statements of understanding. Scaffolding generalizations supports the process of crafting high-quality statements of conceptual understanding. Instruction focuses on the Level 2 generalizations, and Level 3 is provided if more clarification or extension is appropriate. Units of work for a concept-based curriculum mainly focus on Level 2 generalizations, with some Level 3 to support the inductive teaching process.

There are several strategies that can be used to draw generalizations from learners. Examples include using the structured or guided inquiry task or utilizing a graphic organizer that asks for the main ideas and concepts of a unit. At the secondary school level, a year-long course may have five or six instructional units. Each unit may have six to eight generalizations to encourage guided inquiry.

The next chapter looks at the power of the unit web and planner when designing concept-based curriculum and instruction.

Discussion Questions

1. What is the difference between a principle and a generalization in mathematics?
2. Why do we wish our students to understand principles and generalizations in mathematics?
3. What is the difference between Level 1, 2, and 3 generalizations?
4. What opportunities do you provide for your students to demonstrate and communicate their conceptual understanding?
5. How will you develop your skills at crafting generalizations when planning your units of work?