CHAPTER

1

Setting the Stage

Proof has historically been considered a topic first encountered in high school geometry class, featuring a two-column format of "statements" and "reasons." In this chapter, we present reasoning-and-proving as a set of activities that transcends content and format and is accessible to all middle and high school students. Thinking about reasoning-and-proving more broadly will help you to enhance students' understanding of the mathematics they are learning and their ability to construct valid mathematical arguments. While reading this chapter, we encourage you to consider

- how this broader perspective on reasoning-and-proving could benefit students' understandings of mathematics; and
- what it would take to support students in building the capacity to engage in reasoning-and-proving.

IS REASONING-AND-PROVING REALLY WHAT YOU THINK?

Over the past few years, the mathematics teachers at Hoover High School have been concerned about their students' struggles to think and reason mathematically, which was made salient recently when they reviewed the results of an assessment that featured constructed response items. In general, they found that their students had difficulty explaining why an answer was correct beyond providing a procedural description (i.e., describing *what* they did). The teachers had also noticed that while the students were completing the assessment, they seemed to become quickly frustrated when faced with a task they could not easily and quickly solve. Many of the students' responses were incomplete, and it looked like these students had just given up.

In an effort to improve this situation, all of the teachers in the math department committed to try to engage students in more tasks and activities that emphasize reasoning, justifying, and proving. While the students worked on these problems in small groups, the teachers also worked hard to ask more questions, listen to what students were saying, and to not do so much telling. It has not been easy!

Recently, the algebra teachers have been working on improving their students' abilities to write *proofs*—not the formal two-column variety usually found in geometry—but rather algebraic, visual, and narrative arguments that can be used to explain "why things work" and to verify that something is true and it will work for all cases. In their latest professional learning community (PLC) meeting, they decided to give students the Sum of Three Consecutive Integers task (Figure 1.1). Their students had been working on exploring number theory tasks like this, so the teachers thought this would be a next task for them to do.

Proofs: algebraic, visual, and narrative arguments that explain "why things work" and are true for all cases.

FIGURE 1.1 The Sum of Three Consecutive Integers task.

> Prove whether the following statement is true or false: *The sum of any three consecutive numbers is divisible by 3.*

Teaching Takeaway:

Collaborating with colleagues face-to-face or virtually provides an opportunity to share ideas, get feedback on your work, and, in general, support your efforts to improve instruction.

During the PLC meeting, teachers also agreed to identify things that happen during the lesson that they thought were "interesting" and to try to capture these events in some way (e.g., taking notes on a puzzling strategy, collecting samples of interesting student work, recording exchanges with students that they were troubled by). They also agreed to write short vignettes from these artifacts to share with each other. The teachers planned to discuss and analyze the vignettes below at their next PLC meeting.

WE REASON & WE PROVE FOR ALL MATHEMATICS

Vignette 1: Carly Epson's Algebra Class

When I approached Shonda's desk, I noticed that she had created a set of examples that supported the conjecture. She had even written "All of these sums are divisible by three because the sum of their digits is divisible by 3. So it is true because I can't find one that doesn't work."

2 + 3 + 4 = 9	[9 is divisible by 3]
25 + 26 + 27 = 78	[7 + 8 = 15, which is divisible by 3]
151 + 152 + 153 = 456	[4 + 5 + 6 = 15, which is divisible by 3]
2744 + 2745 + 2746 = 8235	[8 + 2 + 3 + 5 = 18, which is divisible by 3]

I knew immediately that Shonda's string of examples didn't prove the conjecture, but I didn't want to say that. My hope was by asking the question "How do you know it will always work?" she would have to think twice about what she had done and see the limitations in her approach. But as you can see in the following exchange, it didn't have that effect at all!

Ms. Epson:	How do you know it will always work?
Shonda:	It has so far and I can't find one that doesn't work.
Ms. Epson:	But how can you be sure?
Shonda:	I am sure.

She is right—she will never find one that doesn't work, so I certainly didn't want to encourage her to try. But not finding a counterexample doesn't mean it will always work. I wasn't sure what to do next to help her move beyond examples. I told her to keep thinking about how she could convince me. But when I checked back later, she had made no progress.

Vignette 2: Jason Steiner's Algebra Class

When I stopped to check in on Keisha, I observed that she had written the following: "The sum of three consecutive numbers is A + B + C. You can't tell if it is divisible by three or not. There's not enough information."

I decided to start asking her some questions that were intended to move her to a more useful way of representing the three numbers. Here is the gist of the exchange:

|	
Mr. Steiner:	So, do you think the statement is true or false?
Keisha:	You can't tell because you don't have enough information.

Mr. Steiner:	What information do you need?
Keisha:	You need to know what one of the numbers is so you can tell what the others will be.
Mr. Steiner:	But suppose that the first number is *x*. What would the next one be?
Keisha:	*y*?
Mr. Steiner:	How much bigger than *x* is the one that comes after *x*?
Keisha:	1 more.
Mr. Steiner:	1 more. So could you write it as $x + 1$?
Keisha:	I guess so.
Mr. Steiner:	Then what would the next biggest number be?
Keisha:	$x + 2$?
Mr. Steiner:	So can you add those three numbers together?
Keisha:	What three numbers?

I was getting frustrated with Keisha and had no idea what to do next besides telling her what to do. I looked around and noticed that Charles had started on an algebraic solution that resembled the one that I was trying to help Keisha develop, so I suggested that Keisha and Charles share their solutions and decide which one they liked best. In the end, Keisha had a solution that looked like Charles's, but I wondered what she understood about it.

Vignette 3: Barbara Law's Algebra Class

One pair of students, Michael and Marissa, asked if they could go get some tiles. I had no idea what they wanted tiles for, but I told them to help themselves to one of the bins of square tiles. When I checked in with them later, they had arranged the blocks as shown below.
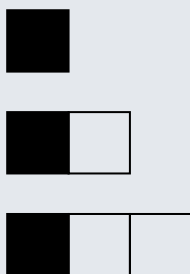
When I asked what the tiles represented, Michael explained, "If the black square is any number, then adding one square to it would be the next number and adding two squares to it would be the one after that." Marissa continued, "If you add the three black squares together, you get three times the number and that is always divisible by three. Then, if you add the three white squares to that number, you will still get a number divisible by three because if you add three to a multiple of three, you get another multiple of three."

I was caught off guard by this approach. I was looking for something more algebraic, and I wasn't sure if this was correct or if it would really count as a proof. I told them it was "interesting" and suggested they try to use algebra to represent the tiles.

Vignette 4: Lynn Baker's Algebra Class

I collected the work from students at the end of class and was not surprised to see that all of the students had clearly defined their variables (x = first number; $x + 1$ = second number; $x + 2$ = third number) because we did this as a class before they wrote their proofs. I was pleased that the students had gone on to show that $x + x + 1 + x + 2 = 3x + 3$. All of the students went on to say that $3x + 3$ had to be divisible by 3 because, as Masey described when she was presenting her group's work, "If you factor out a three, then the threes cancel."

I was really pleased that they all were able to solve the problem and that they got the right answer. Clearly, my students are beginning to understand how to use algebra to prove things!

The work that these algebra teachers are engaged in to improve their students' abilities to think deeply about mathematical concepts and relationships, reason mathematically, and write valid mathematical arguments is commendable. They came together in a PLC to work on their teaching and developed a common goal for improvement that was based on analyzing student data. They chose the same task to implement in their classrooms and gathered artifacts about interesting things that happened during class. They each came away from their class with questions to discuss with each other.

Carly Epson wondered what to do with a student who was convinced that a pattern would always hold true after only testing a few examples. Carly knew that testing examples was not sufficient for arguing the truth of a conjecture, but was unsure what to do next to support Shonda to move to a more generalized argument. Jason Steiner had a similar frustration with Keisha. Barbara Law's student used a pictorial representation to make a generalized argument, and while it seemed convincing, Barbara wondered if this really counted as a proof of the conjecture. And while Lynn Baker felt that her students were well on their way to being able to write algebraic proofs, reading her vignette may have raised questions for you about what her students really knew and were able to do on their own, without her guidance with setting up the variables at the beginning of class.

What questions do these vignettes raise for you about mathematical reasoning-and-proving in the context of middle and high school mathematics?

SUPPORTING BACKGROUND AND CONTENTS OF THIS BOOK

We designed this book to support you as you grapple with how to create learning environments that support middle and high school students to become deeper mathematical thinkers, better reasoners, and more capable of making sound mathematical arguments. We believe that this focus is the central work of mathematics education today and that students benefit when learning mathematics in these kinds of environments. We are not alone in this belief.

The field of mathematics education has long emphasized the importance of reasoning and sense-making in K–12 mathematics for the learning of mathematics with understanding. In 1989, the National Council of Teachers of Mathematics (NCTM) drew our attention to this important aspect of learning mathematics by including *mathematics as reasoning* as one of the standards in its seminal publication of *Curriculum & Evaluation Standards for School Mathematics* (NCTM, 1989). NCTM continued to draw our attention to this focus, and began to emphasize the role of proof by including *reasoning and proving* as an essential component of learning mathematics in *Principles and Standards for School Mathematics* (2000) (see Figure 1.2).

A perusal through NCTM's teacher journals (*Teaching Children Mathematics, Mathematics Teaching in the Middle School,* and *Mathematics Teacher*) as well as publications of a number of books over the past 10 years shows that reasoning-and-proving remains in the forefront of the Council's efforts to improve mathematics education in the United States and around the world. Most notably, NCTM (2009) published *Focus in High School Mathematics: Reasoning and Sense Making.* This document, which broadly defined reasoning as ranging from informal explanations and justifications

FIGURE 1.2 NCTM's reasoning and proving standard.

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof. (p. 56)

Source: NCTM, 2000

to formal deduction or proof, argued "reasoning and sense making should occur in every mathematics classroom every day" (p. 5).

Other standards documents have also advocated for the importance of reasoning-and-proving, with the most recent being the *Common Core State Standards—Mathematics* (CCSSM) (National Governors Association & Council of Chief State School Officers, 2010). CCSSM contains eight ***Standards for Mathematical Practice (SMPs),*** which are intended to guide the types of experiences that students need to have while learning the mathematical content standards outlined in the remainder of the document (see Figure 1.3).

Standards for Mathematical Practice (SMPs): eight CCSSM standards that are intended to guide the types of experiences that students need to have while learning the mathematical content outlined in the remainder of the document.

FIGURE 1.3 CCSSM's Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. **Reason abstractly and quantitatively.**
3. **Construct viable arguments and critique the reasoning of others.**
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. **Look for and make use of structure**.
8. **Look for and express regularity in repeated reasoning.**

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The bolded SMPs are those that connect directly to students' abilities to reason mathematically and engage in mathematical argumentation, although all of the SMPs are supported in learning environments where mathematical reasoning, sense-making, and argumentation are central to everyday work.

How do these kinds of environments benefit middle and high school students? One compelling argument is that these kinds of learning environments support the development of a habit of mind that is useful in mathematics and beyond. In *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009), the authors argue that reasoning and sense making (which includes making valid mathematical arguments) will "enhance students' development of both content and process knowledge they need to be successful in their continued study of mathematics and in their lives" (p. 7). In particular, reasoning and sense-making skills support informed decision making, promote quantitative literacy, support civic engagement, and position graduates to lead in an increasingly technological economy and workforce (American Diploma Project, 2004; NCTM, 2009). Furthermore, Knuth (2002a) argued that proof (as a form of mathematical justification) can be a powerful tool for the learning of mathematics. Along with actions of engaging in pattern seeking, conjecturing, and writing explanatory proofs, presenting and discussing those proofs publicly benefits students because it "can help demonstrate relationships among areas of mathematics that, to many students, seem unconnected" (p. 489).

By working through this book, you will have the opportunity to more deeply consider the benefits of engaging middle and high school students in reasoning-and-proving through two types of activities:

1. Analyses of narrative cases that feature middle and high school teachers and their students engaged in reasoning-and-proving activities; and

2. Through your implementation of, and reflection on, reasoning-and-proving tasks in your own classrooms.

WHAT IS REASONING-AND-PROVING IN MIDDLE AND HIGH SCHOOL MATHEMATICS?

What does it mean to reason-and-prove in middle and high school mathematics? NCTM (2000) and the CCSSM SMPs (Figures 1.2 and 1.3) provide some guidance. According to the Merriam-Webster

Dictionary (www.merriam-webster.com), *to reason* means to "think in a logical way" and *to prove* means to "establish the existence, truth, or validity of (as by evidence or logic)." We could rewrite these definitions to represent what we mean by reasoning-and-proving in a mathematical context. Instead of defining these terms anew for this book, and in an effort to consolidate the ideas presented in Figures 1.2 and 1.3, we have adopted the phrase *reasoning-and-proving* (Stylianides, 2008b; 2010) to encapsulate this kind of mathematical thinking and argumentation.

Reasoning-and-proving describes the following set of activities: identifying patterns, making conjectures, and providing arguments that may or may not qualify as proofs (Stylianides, 2008b; 2010). In mathematics, the development and validation of new knowledge often passes through several stages, and providing a *proof* is typically the last stage. Earlier stages of mathematicians' work frequently involve exploration of mathematical phenomena to identify and arrange significant observations into meaningful *patterns*. Mathematicians then use those patterns to make *conjectures* and ultimately seek to understand and provide *arguments* about whether and why things work. This progression is represented by the arrows between the stages in Figure 1.4.

This description may infer that the progression through the three stages is linear in nature. In actuality, mathematicians often cycle back to previous stages as they come to understand relationships more deeply, develop counterexamples (which disprove a conjecture), and/or realize that they have not done enough work in the previous stage to move forward. This cyclic progression is represented by the left-facing arrows in Figure 1.4.

Knowing the stages of reasoning-and-proving supports teachers whose students have difficulty providing valid mathematical arguments for a

To Reason: to think in a logical way;

To Prove: to establish the existence, truth, or validity of (www.merriam-webster.com).

Reasoning-and-Proving: the set of activities that includes identifying patterns, making conjectures, and providing arguments that may or may not qualify as proofs.

FIGURE 1.4 Three stages of reasoning-and-proving.

identifying a pattern → making a conjecture → providing arguments (which may or may not qualify as proofs)
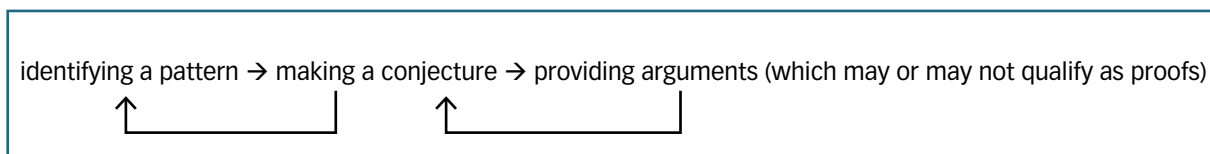
Teaching Takeaway:

The design and wording of a task matters, and how the task is designed should align with the opportunities you want for your students based on their prior experiences.

Teaching Takeaway:

Mathematicians engage in seeking and identifying patterns, making conjectures, and providing an argument, but not always in a linear manner. Expect that students will also move back-and-forth as they engage in reasoning-and-proving activities.

conjecture. Let's return for a moment to the Hoover High School teachers' vignettes presented at the beginning of this chapter. Shonda and Keisha had a difficult time providing a valid argument for the conjecture that was already stated in the problem. Would they have been more successful if the conjecture about the sum of three consecutive numbers had not been already given to them? We wonder what would have happened if the task instead had opened up the opportunity to explore and look for patterns—if it had read something like, "Explore the sums of three consecutive integers. What patterns do you see? Make a conjecture and then provide an argument that shows that your conjecture is always true." We do not know for sure what would have happened with Shonda and Keisha, but we would hope that an exploration like this would reveal more about the mathematical structure of the sum of three consecutive numbers than was allowed with the task given.

This three-stage conceptualization of reasoning-and-proving transcends mathematical domains (e.g., algebra, geometry) and representational forms (e.g., algebraic, pictorial) and is based in the work of mathematicians as they seek to develop and validate new knowledge. Although this process of developing new mathematical knowledge may appear linear in nature, it often is more complex. For example, it is not uncommon that when doing this kind of work, mathematicians' efforts to justify a conjecture yield a counterexample as opposed to a proof. This result can prompt further exploration of the original mathematical phenomenon in order to come up with and then justify a new conjecture (e.g., Lakatos, 1976). As we stated previously, reasoning-and-proving is likely to involve movement back-and-forth between the activities of identifying patterns, making conjectures, and providing arguments.

Your work in Chapters 2 and 3 will broaden and deepen your understandings of, and abilities to engage in, reasoning-and-proving.

REALIZING THE VISION OF REASONING-AND-PROVING IN MIDDLE AND HIGH SCHOOL MATHEMATICS

At the heart of this book is our desire to support teachers in realizing the vision of reasoning-and-proving playing a central role in learning mathematics in middle and high school. One of the challenges of realizing the vision of reasoning-and-proving set forth in the standards documents referenced earlier is that many mathematics teachers have not had the opportunity to engage in reasoning-and-proving activities as mathematics learners or to consider how to incorporate reasoning-and-proving into their mathematics teaching. Thus, the purpose of this book is to provide opportunities for mathematics teachers, and those studying to be mathematics teachers, to enhance their own understandings of, and instructional practices for, promoting reasoning-and-proving in mathematics classrooms in Grades 6–12.

Consider again the vignettes presented at the beginning of this chapter. Even though the Hoover High School teachers had good intentions for engaging their students in reasoning-and-proving activities, those intentions were not fully realized. Carly did not know how to move Shonda beyond thinking only through numerical examples, and Jason did not know how to support Keisha to think more abstractly. Barbara was impressed with her students' reasoning, but she wasn't sure that what they had done counted as a proof. Lynn was pleased that her students were "getting it," but as a reader, you may have questions about whether she gave away too much when she set up the task by guiding her students through creating expressions for three consecutive numbers. These teachers had a need to learn new ways of thinking about reasoning-and-proving as well as to develop teaching strategies to support their students' engagement in reasoning-and-proving.

Another challenge of realizing this vision of reasoning-and-proving set forth in these standards documents is that the development of proofs has often been treated as a formal process in geometry and in isolation from the other activities. (You might find it interesting to read about the place of proof in the American school mathematics curriculum in the past century, which you can find in G. Stylianides [2008a]). This treatment of proof has been problematic, because it has not afforded students the level of scaffolding that mathematicians are able to use when making sense of mathematics. Knuth (2002a)

wrote about the challenge of meeting the vision set forth by these standards documents and advocated that "adopting a view of proof as a tool for meaningfully learning mathematics, a view underlying the recognitions of the explanatory potential of proofs, gives ways to meet this challenge" (p. 490). This book is intended to help teachers rise to the challenge and put the vision of these standards documents into practice.

Teaching Takeaway:

Justifying why things work mathematically must become commonplace in Grades 6–12 mathematics classrooms. Regularly engaging in reasoning-and-proving activities can make justification a classroom norm.

In order for students to have increased opportunities to engage in reasoning-and-proving activities, classrooms must be transformed so that understanding and justifying why things work as they do become commonplace. At its heart, reasoning-and-proving involves searching a mathematical phenomenon for patterns, making conjectures about those patterns, and providing arguments demonstrating the viability of the conjecture. In addition, learning mathematics through engaging in reasoning-and-proving requires participation in a community of learners where making thinking public, justifying conclusions, and debating with peers are all hallmark practices of mathematicians. These practices cannot be learned in classrooms where teachers demonstrate how to do procedures and students practice applying learned procedures with no emphasis on sense-making.

Principles to Actions: Ensuring Mathematical Success for All (NCTM, 2014) contains eight effective teaching practices that research indicates support students' learning of mathematics in a deep and connected way. In this book, we use the lens of the effective teaching practices articulated in *Principles to Action: Ensuring Mathematical Success for All,* with a particular emphasis on five of the practices:

1. Establish mathematical goals to focus learning,

2. Implement tasks that promote reasoning and problem solving,

3. Use and connect mathematical representations,

4. Pose purposeful questions, and

5. Facilitate meaningful discourse.

Your work in Chapters 4–6 will focus on considering these five teaching practices through a reasoning-and-proving lens.

MOVING FORWARD

Striving to enhance instructional practices and create learning environments where students are doing this kind of mathematical work is challenging and takes time and patience. If you are reading this book, you have already taken the first step in seeking ways to enhance your own teaching practices and, thus, your students' understandings of important mathematics. You might be in the beginning stages of learning to teach mathematics, perhaps enrolled in a teacher education program at a college or university. You might be a practicing teacher, seeking professional learning opportunities that will support new ways to think about teaching and learning of mathematics. You may be seeking information and advice specifically about supporting your students' abilities to engage in the kinds of thinking, reasoning, and justification in a similar way to the teachers at Hoover High School.

Regardless of where you are in your teaching career, the activities in this book are designed to help you seriously consider the role of reasoning-and-proving in teaching secondary mathematics and the ways in which you can build your students' capacity to engage in these processes. The teachers at Hoover High School, introduced at the beginning of this chapter, identified the need to engage students in more activities that would strengthen their reasoning skills. Despite their best intentions, however, these teachers struggled at times to figure out how to help students make progress on the task without telling them what to do and how. By the time you finish reading this book, we hope that you have new insights into what the Hoover teachers were trying to do and suggestions for them regarding how they might respond to the dilemmas that surfaced during their lessons. Our goal is to equip you with the determination to support your students as reasoners-and-provers and the pedagogical toolkit that will make reasoning-and-proving a reality in your classroom.

This book was written for readers to actively engage while learning more about reasoning-and-proving. There are mathematical tasks to do, student work to analyze, and narrative cases to examine. You will get the most out of the book if you **stop and do** the activities as you progress. Toward that end, we encourage you to keep a journal (separate notebook) as you work your way through the book. Although we have included 10 blank note-taking pages at the end of the book, they are meant for on-the-spot note taking rather than being sufficient for all of your work as you progress through the book. In your

journal, you can record your solutions to the mathematical tasks you are asked to do and answers to specific questions raised in the activities, and you can respond to the "Pause and Consider" prompts, which are intended to help you reflect on your learning and consolidate your current thinking at a particular moment in time. Your journal is intended to serve as your personal resource during your journey through the book and as you begin to implement reasoning-and-proving in your own classroom.

Discussion Questions

1. The teachers at Hoover High decided to give their students more opportunities to engage in tasks that focused on reasoning, justifying, and proving. Do all middle and high school students really need to be able to engage in these processes? Why or why not?

2. What opportunities do *your* students currently have to engage in the range of activities associated with reasoning-and-proving (i.e., identifying patterns, making conjectures, providing arguments that may or may not qualify as proofs)?

3. What benefits and challenges would be associated with increasing your students' opportunities to engage in reasoning-and-proving?